Clumps into Voids

Nazeem Mustapha*and Charles Hellaby†

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Department of Mathematics and Applied Mathematics
University of Cape Town
Rondebosch
7701

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Abstract

We consider a spherically symmetric distribution of dust and show that it is possible, under general physically reasonable conditions, for an overdensity to evolve to an underdensity (and vice versa). We find the conditions under which this occurs and illustrate it on a class of regular Lemaître-Tolman-Bondi (LTB) solutions. The existence of this phenomenon, if verified, would have the result that the topology of density contours, assumed fixed in standard structure formation theories, would have to change and that luminous matter would not trace the dark matter distribution so well.

1 Introduction

In the inflationary universe paradigm, it is believed that the observed universe is very nearly flat. The density of baryons — which can be obtained from primordial nucleosynthesis theory — is however very small and this requires that most matter is non-baryonic. Traditional theories of structure formation assert that baryonic matter fell into the high density peaks of dark matter and became luminous forming stars and galaxies. The stationary view, in which matter concentrations remain essentially fixed, may be thought of as being governed by a mapping which preserves extremal points of the density field. It may well be a good approximation if the initial density field is simply amplified by gravitational processing, but when the matter content of the pre- and post-decoupling epochs is viewed from a hydrodynamical point of view as a fluid in high-temperature plasma or quasi-plasma state, one would expect shock waves and other spatial gradients to exist (even if their amplitudes were small). Indeed, large scale inhomogeneities and flows have been shown to be a pervasive influence on the behaviour of the universe on scales of up to 100 Mpc. (Cf. for example [13, 3, 8]). The now undisputed existence of large-scale cosmic flows (on the scale of 15000 kms^{-1}) as has been reported by various authors [16], lends more credence to the idea that perhaps the stationary approximation, used ubiquitously in structure formation, is not as good as assumed. The bulk flow reading of $700 \pm 170 kms^{-1}$ found for all Abell clusters with redshifts less than $15000 kms^{-1}$ strongly excludes any of the popular models with Gaussian initial conditions.

In this context, the Lemaître-Tolman-Bondi (LTB) [10, 19, 2] universe is interesting as one may analytically study the evolution of spherically symmetric inhomogeneities. The discovery of large scale voids and walls in the eighties sparked interest in the LTB model as a means of investigating these, and other similar, structures (for example [12, 11, 17]).

*email: nazeem@dyson.mth.uct.ac.za
†email: cwh@maths.uct.ac.za

of anisotropies in the CMB has been studied using LTB models numerically. The results have been that a large part of the temperature anisotropies in the background radiation (the dipole component) may be completely due to large scale structures, but leave open the origin of other sources (for example quadrupole) as truly cosmological. Also worthy of mention is the work done by Lake and Pim [9, 14].

These studies concentrated on the description and feasibility of spherical inhomogeneities, and were not too concerned with determining under what conditions structures could change radically with evolution. Here we intend to initiate analytical studies on this topic.

At the centre of symmetry of an LTB universe, we must generically have a position of extreme density. Thus at the centre it is not feasible to study the question of density waves, *per se*, since a wave is defined by the fact that a maximum (or minimum) moves at some velocity away from the worldline. But at the centre we can ask the question: '*under what conditions will a density maximum evolve into a density minimum or vice versa?*'. This is a first step towards a study of cosmic flows in this model, since if this question can be answered in the affirmative, then it would naturally follow that in some region around the centre over the time elapsed a maximum (or minimum) has to be traveling away from the centre.

If physical, this would raise questions about the validity of the standard model of structure formation. It is particularly important to COBE analyses where the data (for example, hot spots and cold spots corresponding to under- and over-densities respectively) on the last scattering surface is 'transferred' to the current epoch by use of a function which does not assume that the peaks in the matter distribution may change to troughs.

2 Preliminaries and Programme

We are interested in whether the profile of a density inhomogeneity can change significantly with evolution. Specifically, we want to know whether a central maximum in density can evolve into a central minimum, or vice versa. For our investigation we use the simplest inhomogeneous cosmological solution to the Einstein field equations, the LTB model.

This universe model is spherically symmetric, but in general radially inhomogeneous. Space-time is described by a four-dimensional continuum filled by an irrotational perfect fluid with a dust equation of state. We may choose the natural coordinate system labelled by $\{x^a\}_{a=0}^3 = \{t, r, \theta, \phi\}$ suggested by the spherical symmetry. The coordinates are assumed to be comoving with the particles. This allows a definition of a fluid velocity $u^a = \frac{dx^a}{dt}$ such that $u^a = \delta^a_0$ and $u_a u^a = -1$, which would mean that time coordinate t is also cosmic time. For an ideal fluid with mass density ρ and vanishing pressure (dust), the energy-momentum tensor has the form $T^{ab} = \rho u^a u^b$. The conservation of energy-momentum ${T^{ab}}_{;b} = 0$ confines the dust to geodesics and also implies that the mass of any portion of the fluid is conserved through the twice contracted Bianchi Identities .

The metric in synchronous comoving coordinates can be written as

$$ds^2 = -dt^2 + \frac{(R')^2}{1+2E}dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

where $R = R(t, r,)$ acts as a transverse scale factor for individual comoving particles, and $E = E(r)$ is an arbitrary function of the integration which has a dynamic as well as a metric geometric role. R is also the areal radius, that is $4\pi R^2$ describes the surface area of the sphere at comoving radius r at any time t and thus $R(t, r) \geq 0$.

The expression for the invariant energy density, $\rho = \rho(t, r)$, is obtained from the tt field equation:

$$8\pi\rho = \frac{2\,M'}{R^2\,R'} \tag{2}$$

where $M = M(r)$ is another arbitrary function.

$$\frac{1}{2}\left(\frac{\dot{R}}{R}\right)^2 = \frac{M}{R^3} + \frac{E}{R^2}\,. \tag{3}$$

We define a 'scale radius', $p(r)$, and 'scale time', $q(r)$, for non-parabolic models as follows:

$$p(r) = \frac{M}{\pm E} \tag{4}$$

and

$$q(r) = \frac{M}{\sqrt{\pm\,(2E)^3}} \tag{5}$$

which may be viewed as alternative variables to M and E. We (for convenience) may sometimes mix these four variables in the equations below. The '$+$' is applicable in hyperbolic models and the '$-$' is used for elliptic models. In a re-collapsing model, the areal radius at maximum expansion is given by $p(r)$ and the time from creation to destruction is $\pi\, q(r)$.

The Friedmann-like equation can be solved (see for example [2]) in terms of parameter $\eta = \eta(t, r)$. For a non-empty universe, $M \neq 0$, there are three solutions to (3):

$$R = \frac{p}{2}\phi_0, \qquad \xi = \frac{2\,(t - t_B)}{q} \tag{6}$$

where

$$\phi_0 = \begin{cases} \cosh(\eta) - 1, \\ (1/2)\eta^2, \\ 1 - \cos(\eta), \end{cases} \qquad \xi = \begin{cases} \sinh(\eta) - \eta, & E > 0 \\ (1/6)\eta^3, & E = 0 \\ \eta - \sin(\eta), & E < 0 \end{cases} \tag{7}$$

where $t_B = t_B(r)$ is a third arbitrary function. The solutions (6,7) have the same evolution as the corresponding FLRW dust solutions , but with spatially variable M, E and t_B. In contrast to the FLRW models, however, it is quite possible to have all three types of evolution in the same model [6]. In the homogeneous FLRW case – $\rho = \rho(t)$ only – the requirement that η be independent of r at all times in (6, 7) implies that

$$t_B = \text{constant}, \qquad M \propto |E|^{3/2}\,. \tag{8}$$

The function $t_B(r)$ is the 'bangtime function'. Individual shells of matter need not all emanate from one single bang event, but originate at different times as determined by $t_B(r)$. The gradient of t_B generates the decaying modes of the perturbation to an FLRW background [18].

$M(r)$ is the effective gravitational mass within r.

The local geometry is determined by $E(r)$, as is evident from its appearance in the metric, and in fact this function determines the 'embedding angle' [7]. Also if we compare equation (3) with the Newtonian analogue of a dust cloud we see that E also acts as an energy potential; that is, locally hyperbolic, parabolic and elliptic regions occur when $E(r) > 0$, $E(r) = 0$ and $E(r) < 0$ respectively. Its gradient generates the growing modes of the perturbation to an FLRW background [18].

Our method for this investigation is straightforward. We require that the density be smooth through the origin of our coordinate system. Thus the spatial gradient of the density has to vanish at $R(t, r = 0)$ for all t. This would then impose certain restrictions on the three arbitrary functions $M(r)$, $E(r)$ and $t_B(r)$ and their derivatives for this density profile to hold. The change in concavity of the density profile at the origin is determined by the sign of the second radial derivative of the density at that point.

For the required density profile we need expressions for the spatial gradient R' and second and third radial derivatives, R'' and R''' respectively, explicitly as a sum of a product of functions of r and functions of η. The full expressions are somewhat nasty-looking expressions and are not easily understood without detailed analysis, so we have merely recorded them in appendix A.

2.1.1 Shell Crossings

We will impose regularity conditions on the spacetime; excluding shell crossings in particular. Loosely stated, a shell crossing occurs when an inner spherical shell of matter moves faster than an outer shell and eventually bursts through. A locus of points is formed where $R' = 0$ and $R \neq 0$[1]. Since the Kretschmann scalar $K = R_{abcd}R^{abcd}$ diverges, one may consider this to be a 'true' singularity[2]. In contrast to other studies which utilised the high-density regions created by shell crossings as generators of large-scale structure, we require the spacetime to be regular and thus seek to exclude shell crossings. The necessary and sufficient conditions under which shell crossings do not occur were derived by Hellaby and Lake [6].

2.1.2 Behaviour at the Origin

An origin occurs at $r = 0$ when $R(t, r = 0) = 0$ for all t. On any constant t surface away from the bang or crunch, we require that
(a) the density ρ be finite, positive, and non-zero,

$$\frac{M'}{R^2 R'} \to \kappa\rho_0(t) = \text{const} \in (0, \infty) \tag{9}$$

(b) the Kretschmann scalar be finite

$$K = \frac{48M^2}{R^6} + \frac{32MM'}{R^5 R'} + \frac{12(M')^2}{R^4(R')^2} \to K_0(t) = \text{const} \in (-\infty, \infty) \tag{10}$$

and (c) the evolution at $r = 0$ not be different from its neighbourhood, so that $(t - t_B)$, $\phi_0(\eta)$ and $\xi(\eta)$ go smoothly to a finite limit in $(0, \infty)$ as $r \to 0$. Equation (6) then gives us the following behaviour of the arbitrary functions near the origin

$$\frac{R}{p} = \frac{R(\pm E)}{M} \to S_0(t) = \text{const} \in (0, \infty), \qquad q = \frac{M}{(\pm 2E)^{3/2}} \to q_0 = \text{const} \in (0, \infty) \tag{11}$$

If we assume that $E(r)$ and $M(r)$ are analytic at $r = 0$, so that they can be approximated by polynomials in r, then we can further deduce that, as $R \to 0$,

$$E \propto R^2 \to 0, \qquad M \propto R^3 \to 0 \tag{12}$$

and similarly

$$\dot{R} \propto R \to 0 \tag{13}$$

Although M'/M & E'/E both go as $1/R$, the foregoing gives

$$\frac{q'}{q} = \left(\frac{M'}{M} - \frac{3E'}{2E}\right) \to \text{constant or } 0, \tag{14}$$

Thus we have an FLRW-like origin[3].

2.1.3 The Smooth Central Density Criteria

The fractional spatial gradient of the density is obtained by differentiating the density with respect to R on a constant t-slice. Since R is a physically invariant quantity – the areal radius – this will give us results which are not coordinate dependent. We can take a slice in time in a natural way since the coordinate time t is also proper time for comoving dust in a synchronous metric and so also physically

[1] Regular maxima in the spatial sections also have $R' = 0$, but are not shell crossings [6].
[2] Other opinions are that these are non-physical in the sense that they merely indicate the impropriety of extending a simplified fluid description too far.
[3] If the density were allowed to approach zero at the origin, other limiting behaviours of E & M would be possible.

are invariant too.

Now the transformation between (t, r) and (t, R) where $R = R(t, r)$ obeys

$$\left.\frac{\partial r}{\partial R}\right|_t R' = 1, \qquad \left.\frac{\partial r}{\partial R}\right|_t \dot{R} + \left.\frac{\partial r}{\partial t}\right|_R = 0 \tag{15}$$

so for any function of r, say $F(r) = F(r(R, t))$, we define

$$\left.\frac{\partial F}{\partial R}\right|_t \equiv \partial_R F = \frac{dF}{dr}\left.\frac{\partial r}{\partial R}\right|_t = \frac{F'}{R'}\,.$$

So the energy density on a hypersurface of constant time, equation (2), can be written as

$$8\pi\rho = \frac{2\,\partial_R M}{R^2}\,. \tag{16}$$

From the above equation we find the fractional spatial gradient of the density to be

$$\frac{\partial_R \rho}{\rho} = \frac{\partial_{RR} M}{\partial_R M} - \frac{2}{R} \tag{17}$$

where we have determined that

$$\partial_{RR} M \equiv \left.\frac{\partial^2 M}{\partial R^2}\right|_t = (M'' - (\partial_R M)R'')/(R')^2\,. \tag{18}$$

We require the density to be finite and its gradient to vanish at the origin. Thus, for the required density profile we must have

$$\left.\frac{\partial_R \rho}{\rho}\right|_{r=0} = -\frac{1}{(R')^2}\left.\left[2\,\frac{(R')^2}{R} + R'' - \frac{M''}{M'}\,R'\right]\right|_{r=0} = 0\,. \tag{19}$$

We find that this gives us four conditions on the arbitrary functions and their derivatives. For the details in the hyperbolic case, see appendix B.

Requiring the density to be C^1 at the origin also implies that the bangtime function $t_B(r)$ must be at least C^1 at the origin. For the details see appendix B.

2.2 Evolution of the Second Radial Derivative of the Density

To answer the question raised in the introduction, we need to see what happens to the second radial derivative of the density. We can obtain an expression for this quantity by differentiating equation (16) twice with respect to R on a surface of constant time.

$$\frac{\partial_{RR}\,\rho}{\rho} = \frac{2}{R}\left(\frac{3}{R} - \frac{2\,\partial_{RR} M}{\partial_R M}\right) + \frac{\partial_{RRR} M}{\partial_R M} \tag{20}$$

where $\partial_{RR} M$ is given by (18) and $\partial_{RRR} M$ is defined as

$$\partial_{RRR} M \equiv \left.\frac{\partial^3 M}{\partial R^3}\right|_t = \frac{1}{(R')^3}(M''' - (\partial_R M)R''') - \frac{3(\partial_{RR} M)R''}{(R')^2}\,.$$

A more explicit form of the above for the hyperbolic case can be obtained by substituting for R', R'' and R''' into equation (20). Again, this is a most unpleasant-looking expression and not easily assimilated. Appendix C contains the result we get after the assumption of a flat central density has been included.

A change in the density contrast will depend on whether factors on the right hand side of equation (56) change sign with evolution, or terms of different sign become dominant. In order to simplify, we have assumed that all three arbitrary functions have polynomial behaviour at the origin. In addition, we imposed the smooth origin condition (14), which effectively says that in some neighbourhood of the origin, the spacetime is tangent to a homogeneous model. We now single out the dominant functions of η for early and late times in hyperbolic models, which leads to the following digestible expressions.

For early times, $\eta \to 0$,

$$\begin{aligned} \lim_{\eta\to 0} \left.\frac{\partial_{RR}\,\rho}{\rho}\right|_{r=0} &\simeq \left(\frac{12E}{M'}\frac{1}{\eta}\right)^3 6\sqrt{2E}t_B' \left[\frac{1}{3}\left(\frac{t_B'''}{t_B'} - \frac{M'''}{M'}\right)\right. \\ &+ \left.\left(\frac{M''}{M'} - \frac{t_B''}{t_B'}\right)\left(\frac{M''}{M'} - \frac{11}{9}\frac{M'}{M}\right) + \frac{4}{9}\frac{M'^2}{M^2}\right]\,; \end{aligned} \tag{21}$$

with R' at early times given by

$$R'\big|_{\eta\to 0} \simeq \frac{1}{12}\frac{M'}{E}\eta\,.$$

For late times we note that $\sinh\eta \to \cosh\eta$, $\cosh\eta - 1 \to \cosh\eta$ where $\cosh\eta \to e^{\eta}\,/\,2$ as $\eta \to \infty$. Therefore for large η, and at the origin,

$$\begin{aligned} \lim_{\eta\to\infty} \left.\frac{\partial_{RR}\,\rho}{\rho}\right|_{r=0} &\simeq \left(\frac{8E^3}{ME'^2}\frac{1}{\cosh\eta}\right)^2 \left[\frac{1}{4}\left(\frac{M'''}{M'} - \frac{E'''}{E'}\right)\frac{E'^2}{E^2} - \frac{1}{2}\left(\frac{E'}{E}\frac{M''}{M'} - \frac{E''}{E}\right)\frac{M''}{M}\right. \\ &- \left(\frac{1}{2}\frac{E'^2}{E^2} - \frac{7}{9}\frac{E'}{E}\frac{M'}{M} + \frac{1}{9}\frac{M'^2}{M^2}\right)\frac{M''}{M} - \frac{5}{18}\frac{M'^2}{M^2}\frac{E''}{E} \\ &+ \left.\frac{1}{3}\frac{M'^2}{M^2}\left(\frac{19}{12}\frac{E'^2}{E^2} - \frac{16}{9}\frac{M'}{M}\frac{E'}{E} + \frac{4}{9}\frac{M'^2}{M^2}\right)\right]\,, \end{aligned} \tag{22}$$

where R' at late times is given by

$$R'\big|_{\eta\to\infty} \simeq \frac{1}{4}\frac{ME'}{E^2}\cosh\eta\,.$$

To what extent can we tailor the evolution of ρ by choosing the 3 arbitrary LTB functions? We recall that there are four restrictions on the nine quantities E, E', E'', M, M', M'', t_B, t_B' and t_B'' given by equations (48)-(51) ensuring a flat central density. This leaves us with the freedom to fix five of the above at $r = 0$. In addition there are also the conditions for a regular origin (section 2.1.2) and those for no shell crossings [6]. However, the former conditions do not provide any additional choice restrictions once we have specified a flat density at the origin, and the latter are inequality constraints which only limit the range of choice, so are not as severe as the others.

We see in the early time expression (21) there appear terms involving the three derivatives of t_B which do not occur in the late time expression (22). This allows us to fix the early time behaviour of the density. And, likewise, the three derivatives of E occur at late times but not at early times for the relative change in concavity of the density. Thus, in principle, we should be able to independently fix the late time behaviour as well due to this freedom. In effect, we have sufficient freedom to model the density as being overdense initially, and subsequently evolving to an underdense state, or indeed, vice versa. Moreover, it is conceivable that the middle time behaviour could be separately specified, as there are still the derivatives of M to play with.

4 Specific Models

We will consider an initial overdensity changing to an underdensity. So, at early times at the origin, we want the concavity to be negative and at late times, positive. We consider 'exact perturbations'[4] of an

[4]This is just a mathematical device. No averaging or matching procedure to define a background FLRW model has been employed — there is no 'gauge problem' in the sense of cosmological perturbations relative to a background.

$$
\begin{aligned}
M(r) &= M_0 r^3(1+\alpha(r)), \quad \alpha(0)=0\,; && (23)\\
2\,E(r) &= r^2(1+\beta(r)), \quad \beta(0)=0\,; && (24)\\
t_B(r) &= \gamma(r), \quad \gamma(0)=0\,. && (25)
\end{aligned}
$$

These ensure the origin conditions of section 2.1.2 are satisfied. However, they do not necessarily satisfy the restrictions imposed by (48)-(51).

We are also preventing shell-crossing singularities from interfering. These conditions are, for $t > t_B$, $R' > 0$ and hyperbolic models $ER/M > 0$,

$$t'_B \leq 0\,, \quad E' > 0 \;\text{ and }\; M' \geq 0\,.$$

For perturbation functions of the form

$$
\begin{aligned}
\alpha(r) &= Ar^a\,, && (26)\\
\beta(r) &= Br^b\,, && (27)\\
t_B(r) &= Cr^c\,, && (28)
\end{aligned}
$$

the requirement of no shell-crossings leads to the following restrictions on the constants A, B and C:

$$
\begin{aligned}
t'_B \leq 0 &\;\Rightarrow\; C \leq 0 && (29)\\
M' \geq 0 &\;\Rightarrow\; A \geq -\frac{3}{(3+a)\,r^a} && (30)\\
E' > 0 &\;\Rightarrow\; B > -\frac{2}{(2+b)\,r^b}\,. && (31)
\end{aligned}
$$

The smooth central density criteria — that is, in this case, (49) and (51) for $t_B(r)$ or (48) and (50) for $M(r)$ and $E(r)$ — impose the following restrictions on A, B and C (Tables 2 and 1). For simplicity, we will only investigate models where a, b and c are (positive) natural numbers.

Table 1 goes here

Table 2 goes here

Choosing a bangtime function $t_B = Cr^2$, substitution into equation (21) shows that the relative concavity of the density at early times can be fixed as negative by choosing C negative, making t_B a decreasing function; in fact,

$$\lim_{\eta\to 0} \left.\frac{\partial_{RR}\,\rho}{\rho}\right|_{r=0} = \frac{160C}{M_0^3}\,.$$

This automatically satisfies the first requirement for no shell crossings to occur as well. A choice of t_B which is of higher power gives $\dfrac{\partial_{RR}\,\rho}{\rho} = 0$ at the origin at early times. Choosing t_B as a linear function results in a vanishing bangtime perturbation as can be seen from Table 1.

We use equation (22) to determine the late time behaviour. The results after use of Table 2 are tabulated below (Table 3).

Table 3 goes here

also have no shell-crossing singularities.

We will illustrate the phenomenon on a model which has quadratic perturbation functions — that is; a, b and c are all equal to two. We choose $A = 1 \times 10^2$, $B = 1 \times 10^{-6}$ and $C = -3 \times 10^{-8}$. Since we are only interested in qualitative results, we may put $M_0 = 1$.

The density profile this specifies is plotted for a sequence of cosmic time (t) values in figures 1-4. The units (cosmological time, length, mass and density units) are converted as follows

$$\begin{array}{lcl} 1 \text{ ctu} & = & 2.005 \times 10^9 \text{ yrs} \\ 1 \text{ clu} & = & 6.146 \times 10^8 \text{ pc} \\ 1 \text{ cmu} & = & 1.285 \times 10^{22} \ M_\odot \\ 1 \text{ cmu/clu}^3 & = & 3.746 \times 10^{-27} \text{ g/cc} . \end{array}$$

Figures 1 - 4 go here

5 Implications and Discussion

We have shown that, for the simplest inhomogeneous cosmologies — the LTB models for spherically symmetric dust — a change from central density maximum to central density minimum (or vice-versa) during the evolution of the inhomogeneity is entirely possible, and a numerical example was presented. Indeed, given that the early and late time limits of the concavity of the central density profile depend on separate arbitrary functions that have no necessary connection, it would be surprising if profile inversions were not common. The models considered are completely physically reasonable for post-decoupling inhomogeneities.

Perhaps the most important implications of this work derive from the existence proof of the possibility of density profile inversion and density waves[5] in the LTB model. In the real universe, which is much more complex than this model, we expect waves to be generic [4]. The crucial element in our investigation is the importance of the bangtime function t_B and its derivatives at early times. We may recall from section 2 that t'_B generates the decaying mode and E' the growing mode to RW perturbations. The overdensity occurs at early times because we choose t_B in such a way that the result is an overdensity and, in a similar fashion, the underdensity occurs at a late time because we choose E such that it gives that particular type of density profile. Perhaps the reason why the effect obtained here has not been discussed before is because most studies consider linearised perturbations which have the ultimate effect of neglecting the decaying mode.

The most common model of structure formation assumes Cold Dark Matter (CDM) with a Harrison-Zel'dovich spectrum of initial perturbations. Observations indicate that CDM predictions on large scales and small scales are incompatible. In particular, standard CDM has trouble reproducing the large velocity dispersion of luminous matter from the stationary standpoint. Realising the fairly universal failings of standard structure formation theories to explain bulk flow statistics, one might argue that there is some fundamental assumption that must be re-evaluated unless there is a radically different process responsible for structure in the universe. It seems natural to ask if our results might go at least some way in solving these problems.

In the linear theory of structure formation, the topology of density contour surfaces does not change. No links are formed and no chains are broken — the genus of the surface is unchanged since the process is continuous. When nonlinearity is important, the genus of the surfaces evolves as clumps and bubbles form. Even though the statistics today may be non-Gaussian, their structure today will vary depending on the Gaussianity of the initial distribution. However, the way that this occurs will be different if the density profile inverts and if density waves are present, since no longer will overdense regions simply grow monotonically. There will be an interaction of spatial and temporal density gradients. This

[5]as has been numerically discovered previously in many studies (in LTB and related models) on large scale structures, mentioned in the introduction.

formation is to be obtained. The change to the topology of the constant density contours, comparing density waves and no-density waves scenarios, should be examined.

It should be clarified that the density waves discussed here are due to motion of the density maximum through the comoving frame, so that a galaxy that is in the density peak at one time may be outside it at a later or earlier time. This effect may be superimposed on the galaxy flow.

A direct effect of this work is its implications for the transfer function used ubiquitously in standard structure formation theories whereby luminous matter congregates in the peaks of the underlying mass distribution[6]. These peaks do not move; in the sense that they remain attached to the same world line as time evolves. The only change that happens is the infall of matter about these peaks so that the density contrast increases. There is spatial flow of matter, but the spatial distribution of extrema of the initial density field remains invariant. This invariance is broken when density profile inversions occur. The effect is to (amongst other things) change the form of the transfer function. We could reasonably speculate that the transfer function becoming more complicated may perhaps allow one to take a standard scale invariant spectrum and fit it to small, large and intermediate constraints.

Another way of viewing this is that we may not be able to rely on luminous matter being an accurate tracer of total cosmic density, since the density peaks that triggered galaxy formation may have moved on. Similarly, the velocity imparted to forming galaxies may no longer be that of the unseen matter component. The result of the gravitational interaction of the luminous and dark components may be observed flows towards regions which do not seem to be density concentrations. However, such two-component effects are beyond the present study.

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We find

$$\frac{\partial R}{\partial r} = \frac{p}{2}u\phi_0 + \frac{p}{2}\frac{d\phi_0}{d\eta}\,1/\left(\frac{d\xi}{d\eta}\right)\frac{\partial \xi}{\partial r}$$

where ϕ_0 and ξ are given by (7), and $u(r)$ has been defined as

$$u \equiv (\ln p)' = \frac{M'}{M} - \frac{E'}{E}\,. \tag{32}$$

After some manipulation we obtain

$$\frac{\partial R}{\partial r} = \frac{p}{2}v\phi_2 - \frac{p}{q}\,{t_B}'\phi_1 + \frac{p}{2}u\phi_0 \tag{33}$$

where ϕ_1 to ϕ_9 are given in (38)-(46), and $v(r)$ is defined by

$$v \equiv \left(\ln\frac{1}{q}\right)' = \frac{3E'}{2E} - \frac{M'}{M}\,. \tag{34}$$

We proceed in a similar fashion to obtain an expression for the second radial derivative:

$$\begin{aligned}\frac{\partial^2 R}{\partial r^2} \;&=\; \frac{p}{2}v^2\phi_5 - \frac{p}{q}\,{t_B}'v\phi_4 + \frac{2\,p\,({t_B}')^2}{q^2}\phi_3 + \frac{p}{2}\,(v' + 2uv)\phi_2 \\ &-\; \frac{p}{q}\,{t_B}'\,(w+u)\,\phi_1 + \frac{p}{2}\,(u' + u^2)\phi_0\end{aligned} \tag{35}$$

where

$$w(r) \equiv \left(\ln\frac{p}{q}\,{t_B}'\right)' = \frac{E'}{2E} + \frac{{t_B}''}{{t_B}'}\,. \tag{36}$$

And for the third derivative

$$\begin{aligned}\frac{\partial^3 R}{\partial r^3} \;&=\; \frac{p}{2}v^3\phi_9 - \frac{p}{q}\,{t_B}'\,v^2\phi_8 + \frac{2\,p\,({t_B}')^2}{q^2}v\phi_7 - 4p\left(\frac{{t_B}'}{q}\right)^3\phi_6 \\ &+\; \frac{3}{2}\,pv\,(uv + v')\phi_5 - \frac{3p}{2q}\,{t_B}'\,(v' + uv + wv)\phi_4 \\ &+\; \frac{6\,p\,({t_B}')^2}{q^2}\,w\,\phi_3 + \frac{p}{2}(3u^2v + 3uv' + 3u'v + v'')\phi_2 \\ &-\; \frac{p}{2q}\,{t_B}'\,(v' + 2w' + 4u' + 2w^2 + 2uw + 2u^2 + uv - wv)\phi_1 \\ &+\; \frac{p}{2}(u^3 + 3uu' + u'')\phi_0\,.\end{aligned} \tag{37}$$

The above derivatives of R have been expressed in terms of u, v and w because if written in terms of M, E and t_B the expressions become a bit messy and are not very useful in that form at this stage. Quantities determined later will be expressed in terms of the latter variables when appropriate.

The various functions of η used above are

$$\phi_1(\eta) \;=\; \frac{\sinh\eta}{\phi_0} \tag{38}$$

$$\phi_2(\eta) \;=\; \sinh\eta\,\frac{\xi}{\phi_0} \tag{39}$$

$$\phi_3(\eta) \;=\; -\left(\frac{1}{\phi_0}\right)^2 \tag{40}$$

$$\phi_4(\eta) \;=\; \phi_1 - 2\frac{\xi}{\phi_0^2} \tag{41}$$

$$\phi_5(\eta) \;=\; \phi_2 - \frac{\xi^2}{\phi_0^2} \tag{42}$$

$$\phi_6(\eta) = 2\frac{\ }{{\phi_0}^4} \qquad (43)$$

$$\begin{aligned}
\phi_7(\eta) &= 3\phi_3 + 6\sinh\eta\frac{\xi}{{\phi_0}^4} && (44)\\
\phi_8(\eta) &= \phi_1 - \frac{6\xi}{{\phi_0}^2} + 6\sinh\eta\frac{\xi^2}{{\phi_0}^4} && (45)\\
\phi_9(\eta) &= \phi_5 + 2\sinh\eta\frac{\xi^3}{{\phi_0}^4} - \frac{2\xi^2}{{\phi_0}^2} = \phi_2 - \frac{3\xi^2}{{\phi_0}^2} + 2\sinh\eta\frac{\xi^3}{{\phi_0}^4} && (46)
\end{aligned}$$

B The smooth central density criteria

We want the density at the origin to be flat at all times.

We substitute R' and R'' into equation (19) to obtain restrictions on the arbitrary functions $E(r)$, $M(r)$ and $t_B(r)$ for (19) to hold.

We find that

$$\begin{aligned}
&\frac{1}{(R')^2}\left[\frac{p}{2}v^2\left(\phi_5 + 2\frac{{\phi_2}^2}{\phi_0}\right) - \frac{p\,t_B'}{q}\,v\,\left(\phi_4 + 4\frac{\phi_1\phi_2}{\phi_0}\right) + \right.\\
&\frac{2\,p\,(t_B')^2}{q^2}\left(\phi_3 + 2\frac{{\phi_1}^2}{\phi_0}\right) - \frac{p}{2}\left(\frac{M''}{M'}v - 6uv - v'\right)\phi_2 -\\
&\left.\frac{p\,t_B'}{q}\left(w + 5u - \frac{M''}{M'}\right)\phi_1 - \frac{p}{2}\left(\frac{M''}{M'}u - 3u^2 - u'\right)\phi_0\right] \qquad (47)
\end{aligned}$$

must vanish at the origin. Here the ϕ_i are all functions of η and are defined along with $u(r)$, $v(r)$ and $w(r)$ in appendix A. Since the functions of parameter time η are linearly independent of each other, it follows that each of the terms in equation (47) must vanish separately.

From the first term we get

$$\left.\frac{1}{(R')^2}p\,v^2\right|_{r=0} = 0 \qquad (48)$$

whilst the third gives

$$\left.\frac{1}{(R')^2}p\left(\frac{t_B'}{q}\right)^2\right|_{r=0} = 0\,. \qquad (49)$$

Note that the constraint arising from the second term is satisfied if the first (48) and third (49) constraints are. The fourth term combined with equation (48) expands to

$$\left.\frac{1}{(R')^2}p\mathcal{A}\right|_{r=0} = 0\,, \qquad \mathcal{A} \equiv \frac{E'M''}{EM'} - \frac{1}{2}\left(\frac{E'}{E}\right)^2 - \frac{E''}{E} \qquad (50)$$

The fifth term combined with the second in equation (47) yields

$$\left.\frac{1}{(R')^2}p\frac{t_B'}{q}\mathcal{B}\right|_{r=0} = 0\,, \qquad \mathcal{B} \equiv \frac{M''}{M'} - 2\frac{M'}{M} - \frac{t_B''}{t_B'}\,. \qquad (51)$$

The last term produces a condition equivalent to equation (48) combined with (50).

We show now that the requirement of having the density smooth through the origin, leading to the constraints in appendix B, implies that the bangtime function $t_B(r)$ must have vanishing first spatial derivative at the origin. We prove this using the coordinate choice $R' \sim 1 \Rightarrow R \sim r$. As before, we can take the origin to be at $r = 0$ without loss of generality. In these coordinates, the assumption of analytic arbitrary functions near $r = 0$ gives $p = M/(\pm E) \sim r^1$, $q = M/(\pm E)^(3/2) \sim r^0$, since $M \sim r^3$ and $E \sim r^2$. The relation which is of importance to us here is equation (51). It says that

$$\left.\frac{1}{(R')^2}p\frac{t_B'}{q}\left(\frac{M''}{M'} - 2\frac{M'}{M} - \frac{t_B''}{t_B'}\right)\right|_{r=0} = 0\,. \qquad (52)$$

$$\left(\frac{M''}{M'} - 2\frac{M'}{M}\right) \sim \frac{-4}{r} \tag{53}$$

Therefore, either $t_B' \sim r^{-4}$ to make $\mathcal{B}$ zero, or $\mathcal{B} = (M''/M' - 2M'/M - t_B''/t_B')$ diverges as $1/r$ or faster. In the former case we get $t_B \sim r^{-3}$, which is not reasonable — either the universe is infinitely old at the origin only, or it will not emerge from the bang for an infinite time. In the latter case $p(M''/M' - 2M'/M)$ is constant, so we require

$$t_B' p\left(\frac{M''}{M'} - 2\frac{M'}{M}\right) \to 0 \qquad \text{and} \qquad p t_B{}'' \to 0 \tag{54}$$

which, near $r = 0$, implies

$$t_B \sim r^c\,, \qquad c > 1\,. \tag{55}$$

C The Relative Concavity of the Density

With repeated application of equations (48)-(51) ensuring a smooth central density (in particular, taking the bangtime derivative to be vanishing at the origin); and using the variables defined by equations (4), (5), (34), (50) and (51), we find a greatly expanded form of equation (20).

$$\begin{aligned}
&\frac{\partial_{RR}\,\rho}{\rho} \times (R')^4\bigg|_{r=0} \\
= \;& \left[\frac{4\,p^2 v^3}{3}\frac{M'}{M}\right]\left(\phi_9\phi_0 + \phi_5\phi_2 - \frac{8\phi_2^3}{\phi_0}\right) \\
+ \;& \frac{p^2}{4}\left[v^2\left(\frac{M'''}{M'} - \frac{E'''}{E'}\right) + v\frac{M'}{M}\left(\frac{M'''}{M'} - \frac{E'''}{E'}\right)\right. \\
& + \frac{13\,M'\,\mathcal{A}v}{2\,M} - \frac{9\,M''\mathcal{A}v}{2\,M'} - \frac{M''\,v^3}{M'} - \frac{2\,M''\,v^2}{M} \\
& \left. - \frac{M''\,M'\,v}{M^2} - \frac{29\,M'v^3}{9\,M} + \frac{26\,M'^2v^2}{9\,M^2} + \frac{4\,M'^3v}{9\,M^3}\right]\phi_2^2 \\
+ \;& \frac{2\,p^2 t_B'}{q}\left[-v\left(\frac{M'''}{M'} - \frac{t_B'''}{t_B'}\right) - v\left(\frac{M'''}{M'} - \frac{E'''}{E'}\right) - \frac{M'}{M}\left(\frac{M'''}{M'} - \frac{E'''}{E'}\right)\right. \\
& \left. + \frac{3\,M''\,\mathcal{B}v}{M'} - \frac{19\,M'\,\mathcal{B}v}{3\,M} + \frac{8\,M''\,v}{M} + \frac{M''\,M'}{M^2} - \frac{94\,M'^2v}{9\,M^2} - \frac{4\,M'^3}{9\,M^3}\right]\phi_1\phi_2 \\
+ \;& \frac{p^2}{4}\left[\frac{3\,M'\,\mathcal{A}v}{2\,M} + \frac{17\,M'v^3}{9\,M} + \frac{7\,M'^2v^2}{9\,M^2}\right]\phi_5\phi_0 - 2\,\frac{p^2 t_B'}{q}\left[\frac{7\,M'^2v}{9\,M^2}\right]\phi_4\phi_0 \\
+ \;& \frac{p^2}{4}\left[\frac{4}{3}\left(v + \frac{M'}{M}\right)\left(\frac{M'}{4\,M} - v\right)\left(\frac{M'''}{M'} - \frac{E'''}{E'}\right)\right. \\
& - \frac{3\,M''\,\mathcal{A}}{2\,M} + \frac{5\,M'^2\mathcal{A}}{6\,M^2} + \frac{6\,M''\mathcal{A}v}{M'} - \frac{35\,M'\mathcal{A}v}{3\,M} \\
& + \frac{4\,M''\,v^3}{3\,M'} + \frac{7\,M''\,v^2}{3\,M} + \frac{2\,M''\,M'\,v}{3\,M^2} - \frac{M'^2M''}{3\,M^3} \\
& \left. - \frac{38\,M'v^3}{27\,M} - \frac{47\,M'^2v^2}{9\,M^2} - \frac{4\,M'^3v}{27\,M^3} + \frac{4\,M'^4}{27\,M^4}\right]\phi_2\phi_0 \\
+ \;& \frac{2\,p^2 t_B'}{3\,q}\left[\left(2\,v - \frac{M'}{M}\right)\left(\frac{M'''}{M'} - \frac{t_B'''}{t_B'}\right) + 2\left(v + \frac{M'}{M}\right)\left(\frac{M'''}{M'} - \frac{E'''}{E'}\right)\right. \\
& + \frac{3\,M''\,\mathcal{B}}{M} - \frac{11\,M'^2\mathcal{B}}{3\,M^2} - \frac{6\,M''\,\mathcal{B}v}{M'} + \frac{79\,M'\,\mathcal{B}v}{6\,M} - \frac{16\,M''\,v}{M} \\
& \left. + \frac{4\,M''\,M'}{M^2} + \frac{65\,M'^2v}{3\,M^2} - \frac{46\,M'^3}{9\,M^3}\right]\phi_1\phi_0
\end{aligned}$$

$$
\begin{aligned}
+ \quad & \frac{\quad}{4}\Bigg[\frac{\quad}{9}\left(v+\frac{\quad}{M}\right)\left(2v-\frac{\quad}{M}\right)\left(\frac{\quad}{M'}-\frac{\quad}{E'}\right) \\
& +\frac{M''\,\mathcal{A}}{M}-\frac{5\,M'^2\mathcal{A}}{9\,M^2}-\frac{2\,M''\,\mathcal{A}v}{M'}+\frac{38\,M'\,\mathcal{A}v}{9\,M} \\
& -\frac{4\,M''\,v^3}{9\,M'}-\frac{2\,M''\,v^2}{3\,M}+\frac{2\,M'^2M''}{9\,M^3}+\frac{76\,M'v^3}{81\,M} \\
& +\frac{50\,M'^2v^2}{27\,M^2}-\frac{8\,M'^3v}{81\,M^3}-\frac{8\,M'^4}{81\,M^4}\Bigg]\phi_0^2
\end{aligned}
\tag{56}
$$

$c = 1$	$C = 0$
$c \geq 2$	no restrictions

Table 1: Restrictions imposed on the perturbation t_B by the requirement of a C^1 central density ($\forall\, c \in {}^+$).

	$a = 1$	$a \geq 2$
$b = 1$	$A = B$	$B = 0$
$b \geq 2$	$A = 0$	no restrictions

Table 2: Restrictions imposed on the perturbations to E and M by the requirement of a flat central density ($\forall a, b \in {}^+$).

	$a = 1$	$a = 2$	$a \geq 3$
$b = 1$	$1/2\,(-A/r + A^2)$	$2A$	0
$b = 2$	$-3B$	$2A - 3B$	$-3B$
$b \geq 3$	0	$2A$	0

Table 3: The sign of $(\partial_{RR}\,\rho)\,/\,\rho$ at the origin at late times.

Figure 1: The density profile on a worldline at an early time ($t = 1 \times 10^{-6}$ ctu $\approx$ 2000 yrs). In these figures, the core is taken to be the value of R at the comoving radius $r = 0.04$ and they all use base 10 logs. At this time the value corresponds to an overdensity of about 1.7 kpc in diameter which corresponds to the size of a small galaxy today. The units in all the figures are cosmological. The value of the density at the origin is $\rho_0 \approx 2.0 \times 10^{-16}$ g/cc.

Figure 2: The density profile on a worldline at a later time. This diagram and the next one illustrate the change in concavity at the centre, which occurs when the universe was $\approx 2 \times 10^4$ years old. $R \approx 7.9$ kpc and $\rho_0 \approx 2.0 \times 10^{-18}$ g/cc.

Figure 3: The density profile on a worldline at a still later time which, when compared to the previous figure, illustrates the movement of the maximum away from the centre. This is as we expected and shows that a density wave must exist near the origin if the profile inverts on the central worldline. $t \approx 4 \times 10^4$ yrs, $R \approx 12.5$ kpc and $\rho_0 \approx 5 \times 10^{-19}$ g/cc.

Figure 4: The density profile on the worldline today ($t \approx 20$ Gyrs). This corresponds to a void (albeit one with a rather elongated wall) with a diameter of approximately 100 Mpc. $\rho_0 \approx 5 \times 10^{-31}$ g/cc and the maximum density on this diagram is $\rho_m \approx 10^{-3.5}$ cmu/clu$^3 \approx 1.2 \times 10^{-30}$ g/cc.